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Discussion

Comments on "investigation of the properties of the period for the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$"

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Abstract

Two Lindstedt–Poinaré perturbation-based methods are used to solve the nonlinear differential equation of a nonlinear oscillator having the square of the angular frequency quadratic dependence on the velocity. Mickens published two interesting papers [J. Beatty, R.E. Mickens, A qualitative study of the solutions to the differential equation $\ddot{x} + (1 + \dot{x}^2)x = 0$, *Journal of Sound and Vibration* 283 (2005) 475–477; R.E. Mickens, Investigation of the properties of the period for the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$, *Journal of Sound and Vibration* 292 (2006) 1031–1035] about this oscillator and by using the harmonic balance method he found that the approximate frequency is not defined for amplitudes of magnitude equal to or larger than two. We show that these standard perturbation methods work better than the harmonic balance method. In particular, the modified Lindstedt–Poincaré method works well for the whole range of oscillation amplitudes, and excellent agreement of the approximate frequency with the exact one has been demonstrated and discussed.

The present authors take this opportunity (i) to congratulate with the authors of Refs. [1,2] for their interesting study, and (ii) to add some interesting results about the angular frequency, $\omega(A)$, as a function of the initial amplitude A, obtained by means of a standard perturbation procedure [3], not already included in Refs. [1,2].

Mickens et al. had published two very interesting papers [1,2] about the nonlinear oscillator

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{1}$$

with initial conditions $x(0) = A$ and $\dot{x}(0) = 0$. Eq. (19) corresponds to a generalized conservative system [3]. In Ref. [2], the author states that this oscillator "has the interesting feature that its angular frequency, $\omega(A) = 2\pi/T(A)$, is singular or not defined at finite values of A when standard perturbation procedures [1,4–6] are used to calculate $\omega(A)$". He also demonstrated following a very interesting procedure that $\omega(A)$ has not singularity for $0 \leqslant A < \infty$, and he pointed out that "the restrictions on the range of applicable A values for $\omega(A)$, obtained by use of various perturbation procedures [1,4–6] are only indications of their limitations for calculating the angular frequency for the nonlinear oscillator given by Eq. (1)". Finally, at the end of Ref. [1], the authors

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pointed out that ‘‘the observed singularities occurring in the various methods to calculate approximate solutions for Eq. (1) are therefore artifacts of the perturbations methods and thus indicate limitations on these techniques’’. A similar sentence to this one was included at the end of Ref. [2]. However, neither in Ref. [1] nor in Ref. [2] the authors did not present an approximate expression for the angular frequency for the oscillator of Eq. (1) by using perturbation methods. They calculated the angular frequency, $\omega(A)$, by means of the first-order harmonic balance method [4] and they obtained the following expression [1,2]:

$$\omega(A) = \frac{2}{\sqrt{4 - A^2}}, \tag{2}$$

which is not defined for amplitudes of magnitude equal to or larger than two in value. In Refs. [1] and [2], the authors did not calculate $\omega(A)$ by means standard perturbation methods but they concluded that the angular frequency obtained by using standard perturbation methods is also singular. It should also be indicated that we have calculated $\omega(A)$ by means of the second-order harmonic balance method [7] and $\omega(A)$ still has a singularity as a function of the amplitude.

In this paper, we will obtain the angular frequency for the oscillator of Eq. (1) by using the standard Lindstedt–Poincaré perturbation method and we will see that this approximate angular frequency has not any singularity as a function of the amplitude. This technique can be used to construct uniformly valid periodic solutions to second-order nonlinear differential equation of the form [3]

$$\frac{d^2y}{dt^2} + y = \varepsilon F\left(y, \frac{dy}{dt}\right), \quad \varepsilon > 0, \tag{3}$$

where ε is a small positive parameter known as perturbation parameter. If we compare Eq. (1) with Eq. (3) we can see that there is no perturbation parameter in Eq. (1). Then the standard Lindstedt–Poincaré method could not be directly applied to Eq. (1). However, it is possible to construct an artificial perturbation equation by embedding an artificial parameter in Eq. (1). The use of a scaling allows to solve this problem. For small amplitude oscillations we introduce the scaling variable ε as follows:

$$x(t) = \sqrt{\varepsilon}\, y(t), \tag{4}$$

where ε is a small positive parameter and the initial conditions can be written as $y(0) = A/\sqrt{\varepsilon} = a$ and $dy(0)/dt = 0$.

Substitution of Eq. (4) into Eq. (1) gives

$$\frac{d^2y}{dt^2} + y + \varepsilon y\left(\frac{dy}{dt}\right)^2 = 0. \tag{5}$$

The method of Lindstedt–Poincaré is based on the change of the time variable $\tau = \omega t$. The equation of motion (5) thus becomes

$$\omega^2\frac{d^2y}{d\tau^2} + y + \varepsilon\omega^2 y\left(\frac{dy}{d\tau}\right)^2 = 0. \tag{6}$$

The solution of Eq. (6) is assumed in the form

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \varepsilon^3 y_3(\tau) + \cdots \tag{7}$$

and ω^2 is also expanded in powers of the parameter ε,

$$\omega^2 = 1 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \varepsilon^3\alpha_3 + \cdots, \tag{8}$$

where the constants α_1, α_2, α_3, … can be identified by means of no secular terms. Substituting Eqs. (7) and (8) into Eq. (6), and equating the terms with the identical powers of, we can obtain a series of linear equations, and we write only the first four linear equations

$$y'' + y_0 = 0, \tag{9}$$

$$y_1'' + y_1 = -y_0(y_0'')^2 - \alpha_1 y_0'', \tag{10}$$

$$y_2'' + y_2 = -y_1(y_0')^2 - 2y_0y_0'y_1' - \alpha_1 y_0(y_0')^2 - \alpha_1 y_1'' - \alpha_2 y_0'', \tag{11}$$

$$\begin{aligned} y_3'' + y_3 = & -y_2(y_0')^2 - y_0(y_1')^2 - 2y_0y_0'y_2' - 2y_1y_0'y_1' - \alpha_1 y_1(y_0')^2 - \alpha_1 y_2'' \\ & - 2\alpha_1 y_0y_0'y_1' - \alpha_2 y_1'' - \alpha_2 y_0(y_0')^2 - \alpha_3 y_0'' \end{aligned} \tag{12}$$

with the initial conditions

$$y_0(0) = a, \quad y_0'(0) = 0, \tag{13}$$

$$y_n(0) = 0, \quad y_n'(0) = 0, \quad n = 1, 2, 3, \ldots . \tag{14}$$

In Eqs. (9)–(14), the prime stands for differentiation with respect to τ. We choose the value of the coefficient α_n in order to remove any secular term from the perturbation equation of order n and we obtain the coefficients

$$\alpha_1 = \frac{a^2}{4}, \quad \alpha_2 = \frac{5a^4}{128}, \quad \alpha_3 = \frac{5a^6}{1536}. \tag{15}$$

Substitution of Eq. (15) into Eq. (8) gives the approximate angular frequency, ω_{app}, as follows:

$$\omega_{\text{app}}(\varepsilon, a) = \sqrt{1 + \varepsilon\frac{A^2}{4} + \varepsilon^2\frac{5A^4}{128} + \varepsilon^3\frac{5A^6}{1536}}. \tag{16}$$

In terms of the original variable $x = \sqrt{\varepsilon}\,y$, the last expression can be rewritten, using $A = \sqrt{\varepsilon}\,a$, to the form

$$\omega_{\text{app}}(A) = \sqrt{1 + \frac{A^2}{4} + \frac{5A^4}{128} + \frac{5A^6}{1536}}. \tag{17}$$

As we can see, this angular frequency is defined for $0 \leqslant A < \infty$. Note that $\varepsilon a^2 = A^2$ is the true small parameter, not ε. Therefore, this approximation is valid only for small values of the amplitude oscillations A and we can conclude that the standard Lindstedt–Poincaré method gives excellent approximate frequencies only for small values of A^2.

Comparing the approximate period $T_{\text{app}}(A) = 2\pi/\omega_{\text{app}}(A)$ with the exact value of the period calculated numerically, it can be seen that the relative error of the approximate value is less than 1%, 2% and 5% for $A < 2.08$, $A < 2.29$ and $A < 2.65$, respectively. However, this relative error increases quickly for large values of A. For example the relative error is 84% for $A = 10$. For small amplitudes it is possible to do the power-series expansion of the approximate period $T_{\text{app}}(A) = 2\pi/\omega_{\text{app}}(A)$. Doing this gives the result

$$T_{\text{app}}(A) = 2\pi\left(1 - \frac{A^2}{8} + \frac{A^4}{256} + \frac{5A^6}{6144} + \cdots\right). \tag{18}$$

It is important to point out that in Eq. (18) the first four terms are the same as the first four terms of the equation obtained in the power-series expansion of the exact period T for small amplitudes and whose value is [8]

$$T(A) = 2\pi\left(1 - \frac{A^2}{8} + \frac{A^4}{256} + \frac{5A^6}{6144} - \frac{7A^8}{262144} - \frac{133A^{10}}{10485760} + \cdots\right). \tag{19}$$

An important property of the period of this oscillator is its behaviour for small and large amplitudes A. Mickens found that $T \to 2\pi$ when $A \to 0$ and $T \to 0$ when $A \to \infty$ [2]. As we can see, the approximate period in Eq. (18) obtained by applying the standard Lindstedt–Poincaré method shows this behaviour for small as well as for large amplitudes of oscillation. However, it has been recently shown that for large amplitudes of oscillation the exact period $T(A)$ of this oscillator satisfies [8]

$$\lim_{A \to \infty}(AT(A)) = 2\pi, \tag{20}$$

however, from Eq. (17)

$$\lim_{A\to\infty}(AT_{\text{app}}(A)) = 0, \quad \lim_{A\to\infty}\frac{T_{\text{app}}(A)}{T(A)} = 0. \tag{21}$$

This result confirms the fact that the standard Lindsted–Poincaré method is valid only for small perturbation parameters, in this example, for small values of A^2.

The problems appear for higher-order approximations because it is obtained that α_4 is negative

$$\alpha_4 = -\frac{3a^8}{131{,}072} \tag{22}$$

and the new approximate frequency up to this new approximation is

$$\omega_{\text{app}}^{(4)}(A) = \sqrt{1 + \frac{A^2}{4} + \frac{5A^4}{128} + \frac{5A^6}{1536} - \frac{3A^8}{131{,}072}}, \tag{23}$$

which is more accurate than Eq. (17) for small amplitudes; however, it is not defined for amplitudes of magnitude equal to or larger than 12.40 in value. In other words, it presents a similar problem to that of Eq. (2).

It is possible to obtain a more accurate approximate frequency (and period) for this oscillator by using a modified Lindstedt–Poincaré method. In this method, the original parameter ε in Eq. (6) is transformed into a new small parameter δ defined as follows [9]:

$$\delta = \frac{\varepsilon\alpha_1}{1+\varepsilon\alpha_1}. \tag{24}$$

Eq. (24) is essentially the Shohat transformation [10] and Section 2.6 of Mickens' book [6] discusses this transformation and presents its application to the Duffing and van der Pol equations. It is easy to verify that $\delta\to 0$ as $\varepsilon\alpha_1\to 0$ while $\delta\to 1$ as $\varepsilon\alpha_1\to\infty$. From Eq. (24) we have

$$\varepsilon = \frac{\delta}{\alpha_1(1-\delta)} \tag{25}$$

and

$$1+\varepsilon\alpha_1 = \frac{1}{1-\delta}. \tag{26}$$

Then Eq. (8) can be rewritten as follows:

$$\omega^2 = \frac{1}{1-\delta}(1+\delta^2\beta_2+\delta^3\beta_3+\cdots), \tag{27}$$

where [9]

$$\beta_n = \frac{\alpha_n}{\alpha_1^n}(1+\varepsilon\alpha_1)^{n-1}, \quad n = 2,3,4,\ldots \tag{28}$$

are unknown which will be successfully determined in the later perturbation steps.

Substituting Eqs. (25) and (27) into Eq. (6) yields

$$(1-\delta)(1+\delta^2\beta_2+\delta^3\beta_3+\cdots)\frac{d^2y}{d\tau^2} + (1-\delta)^2 y + \frac{\delta}{\alpha_1}(1+\delta^2\beta_2+\delta^3\beta_3+\cdots)y\left(\frac{dy}{d\tau}\right)^2 = 0. \tag{29}$$

Now we expand the solution of Eq. (6) into a power series in the new parameter δ

$$y(\tau) = \bar{y}_0(\tau) + \delta\bar{y}_1(\tau) + \delta^2\bar{y}_2(\tau) + \delta^3\bar{y}_3(\tau) + \cdots \tag{30}$$

$y_n(\tau)$ in Eq. (7) and $\bar{y}_n(\tau)$ in Eq. (30) are different. Substituting Eq. (30) into Eq. (29) and equating the coefficients of the same power of δ, a new set of linear differential equations can be obtained instead of

Eqs. (9)–(12). The usual steps in the Lindstedt–Poincaré method may be applied to solve these perturbation equations and the solution of Eq. (6) can be obtained to any desired order of δ. Following a similar procedure to eliminate secular terms than that used previously, the following results can be obtained:

$$\alpha_1 = \frac{a^2}{4} = \frac{A^2}{4\varepsilon} \tag{31}$$

and

$$\beta_2 = \frac{5}{8}, \quad \beta_3 = \frac{5}{6}, \quad \beta_4 = \frac{1591}{1536}, \quad \beta_5 = \frac{383}{320}. \tag{32}$$

Substitution of Eq. (31) into Eqs. (24) and (26) gives

$$\delta = \frac{A^2}{4 + A^2} \tag{33}$$

and

$$\frac{1}{1-\delta} = 1 + \frac{A^2}{4}, \tag{34}$$

while substitution of Eqs. (32), (33) and (34) into Eq. (27) gives the following new expression for the approximate frequency:

$$\omega_{\text{app}}(A) = \sqrt{1+\frac{A^2}{4}}\sqrt{1 + \frac{5A^4}{128(1+(A^2/4))^2} + \frac{5A^6}{384(1+(A^2/4))^3} + \frac{1591A^8}{393,216(1+(A^2/4))^4} + \frac{383A^{10}}{327,680(1+(A^2/4))^5}}. \tag{35}$$

For higher-order approximations we have obtained that $\beta_6 > 0$ and $\beta_7 > 0$; so we can state that to this new approximation the approximate frequency is defined by $0 \leqslant A < \infty$. Comparing the approximate period $T_{\text{app}}(A) = 2\pi/\omega_{\text{app}}(A)$ with the exact value of the period calculated numerically, it can be seen that the relative error of the approximate value is less than 1%, 2% and 5% for $A < 1.94$, $A < 2.22$ and $A < 3.10$, respectively. Unlike what happened by applying the standard Lindstedt–Poincaré method, the relative error does not grow much for large values of A. For example, it is 4.8% for $A = 10$. For small amplitudes it is possible to expand in the power series the new approximate period $T_{\text{app}}(A) = 2\pi/\omega_{\text{app}}(A)$. Doing this the following result is obtained:

$$T_{\text{app}}(A) = 2\pi\left(1 - \frac{A^2}{8} + \frac{A^4}{256} + \frac{5A^6}{6144} - \frac{7A^8}{262,144} - \frac{133A^{10}}{10,485,760} + \cdots\right). \tag{36}$$

It is important to point out that in Eq. (36) the first six terms are the same as the first six terms of the equation obtained in the power-series expansion of the exact period T for small amplitudes and whose value is given in Eq. (16). The new approximate period satisfies the requirements that $T \to 2\pi$ when $A \to 0$ and $T \to 0$ when $A \to \infty$ and also it has the following behaviour for large amplitudes:

$$\lim_{A\to\infty}(AT_{\text{app}}(A)) = 0.9234(2\pi), \quad \lim_{A\to\infty}(T_{\text{app}}(A)/T(A)) = 0.9234. \tag{37}$$

Therefore, the relative error for this new approximate period is less than 7.7% for large values of the amplitude oscillation, while for the standard Lindstedt–Poincaré method this relative error was 100%. In Fig. 1, we have plotted the percentage error defined as

$$\text{relative error (\%)} = 100\left|\frac{T(A) - T_{\text{app}}(A)}{T(A)}\right| \tag{38}$$

for the approximate periods obtained by means of the standard and the modified Lindstedt–Poincaré perturbation methods considered in this paper. As we can see, the second method provides better results for the whole range of oscillation amplitudes. The essential idea of the Lindstedt–Poincaré method is to expand the square of the unknown frequency ω^2 in power series in terms of ε near the linear frequency 1, however, when ε is large (A^2 for our oscillator) this method is not suitable. The essential idea of this modified Lindstedt–Poincaré method is to expand ω^2 in power series near a new position $1+\varepsilon\alpha_1$ in terms of a new

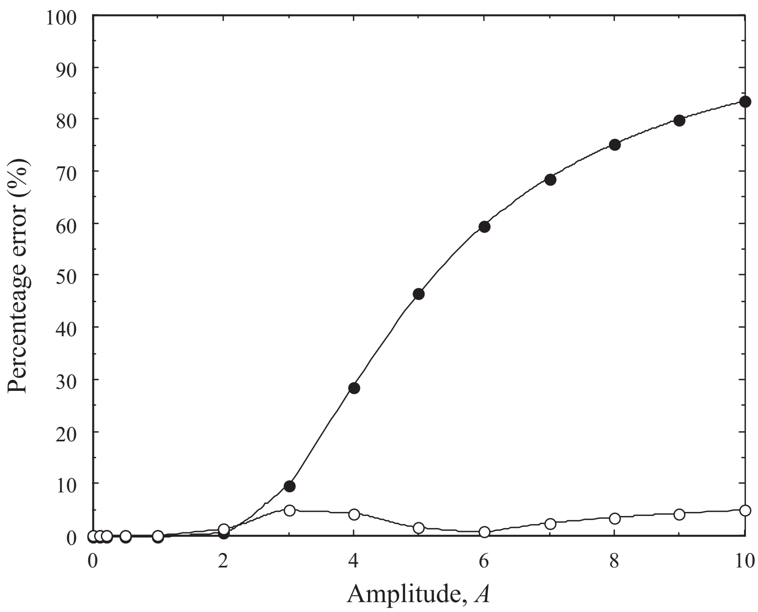

Fig. 1. Percentage error for the approximate periods obtained by means of the standard (●) and the modified (○) Lindstedt–Poincaré perturbation methods considered in this paper. The period is $T_{app}(A) = 2\pi/\omega_{app}(A)$ and in the first case ω_{app} is obtained by Eq. (17) while in the second one ω_{app} is obtained by Eq. (35).

parameter δ, which keeps a small value regardless of the magnitude of ε. Therefore, Eq. (27) will lead to better results than that Eq. (8). In summary, this paper shows how the difficulties arising in Mickens' previous works [1,2] on this oscillator can be resolved using a modified Lindstedt–Poincaré perturbation method.

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